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ON STOCHASTIC NON-HOLONOMIC SYSTEMS*

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A natural mechanical system is considered with ideal stochastic non-holonomic constraints under the action of potential, dissipative, and perturbing forces that depend on random (in general non-normal) paramters satisfying non-linear Ito stochastic differential equations. The corresponding stochastic equations of motion are constructed in Lagrangian and Hamiltonian variables, as well as the equations for finite-dimensional densities and characteristic functions. Stationary one-dimensional distributions are studied in Chaplygin normal stochastic non-holonomic systems. Systematic and fluctuational drift is analysed. The problems of a rolling ball and a rolling convex rigid body on a translationally vibrating horizontal plane are considered.

1. Assume that at each instant of time t the location of the natural non-holonomic stochastic system with ideal constraints is defined by the vector of generalized coordinates $\mathbf{q} = [q_1 \ldots q_n]^T$ and the system velocities are constrained by m conditions of the form

$$\mathbf{a}^\varkappa \mathbf{q}^\cdot + a_0^\varkappa = 0, \quad \varkappa = 1, \ldots, m \tag{1.1}$$

where $\mathbf{a}^\varkappa = \mathbf{a}^\varkappa(\mathbf{q}, \boldsymbol{\pi}, t)$ are n-dimensional row matrices, which are deterministic functions of the generalized coordinates $\mathbf{q}$, the random parameters $\boldsymbol{\pi} = [\pi_1(t) \ldots \pi_k(t)]^T$, and time t; $a^\varkappa_0 = a^\varkappa_0(\mathbf{q}, \boldsymbol{\pi}, t)$ are deterministic functions of the listed variables.

We assume that the random vector $\boldsymbol{\pi}$ satisfies an Ito stochastic differential equation of known form with the corresponding initial conditions:

$$\boldsymbol{\pi}^\cdot = \boldsymbol{\varphi}(\boldsymbol{\pi}, t) + \boldsymbol{\psi}(\boldsymbol{\pi}, t)\mathbf{V}, \quad \mathbf{V} = [V_1(t) \ldots V_l(t)]^T, \quad \boldsymbol{\pi}(t_0) = \boldsymbol{\pi}_0 \tag{1.2}$$

Here $\boldsymbol{\varphi}(\boldsymbol{\pi}, t)$ and $\boldsymbol{\psi}(\boldsymbol{\pi}, t)$ are $k \times 1$ and $k \times l$ deterministic functions, respectively, $\mathbf{V} = d\mathbf{W}/dt$ is the (strict-sense) white noise vector, and $\mathbf{W} = \mathbf{W}(t)$ is an arbitrary stochastic process with independent increments which has zero mean and a finite covariance matrix and is independent of π_0. This process can, in general, be expressed by the formula /1/

$$\mathbf{W}(t) = \mathbf{W}_0(t) + \int_{R^l} \mathbf{c}(\mathbf{x}) \mathbf{P}^c(t, d\mathbf{x}) \tag{1.3}$$

Here $\mathbf{W}_0(t)$ is a Wiener (normal) stochastic process, $\mathbf{c}(\mathbf{x})$ is a vector function (of the same dimension l as the process $\mathbf{W}(t)$) of the l-dimensional vector argument $\mathbf{x}$, and the integral for any $t \geqslant t_0$ is an Ito stochastic integral over the centred Poisson measure $\mathbf{P}^\circ(t, dx)$ which is independent of the process $\mathbf{W}(t)$ and takes independent values on pairwise disjoint sets. The intensity $\boldsymbol{\nu}(t)$ of the process $\mathbf{W}(t)$ is given by the formula

$$\boldsymbol{\nu}(t) = \boldsymbol{\nu}_0(t) + \int_{R^l} \mathbf{c}(\mathbf{x}) \mathbf{c}(\mathbf{x})^T \boldsymbol{\nu}_P(t, \mathbf{x})\, d\mathbf{x} \tag{1.4}$$

where $\boldsymbol{\nu}_0(t)$ is the intensity of the Wiener process $\mathbf{W}_0(t)$ and $\boldsymbol{\nu}_P(t, \mathbf{x})$ is the intensity of the stream of jumps $\mathbf{c}(\mathbf{x})$ of the process $\mathbf{W}(t)$.

We assume that the (n-dimensional) vector of generalized perturbing forces can be represented in the form

$$\mathbf{Q} = \mathbf{Q}(\mathbf{q}, \mathbf{q}^\cdot, \mathbf{V}, \boldsymbol{\pi}, t) = \mathbf{Q}^1(\mathbf{q}, \mathbf{q}^\cdot, \boldsymbol{\pi}, t) + \mathbf{Q}^2(\mathbf{q}, \mathbf{q}^\cdot, \boldsymbol{\pi}, t)\mathbf{V} \tag{1.5}$$

Following Voronets /2/, we obtain an Ito stochastic equation of motion in Lagrange variables. Indeed, since the constraints (1.1) are ideal, we obtain by the D'Alembert-Lagrange principle the equation

$$\sum_{s=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_s^\cdot}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial \Pi}{\partial q_s} + \frac{\partial R}{\partial q_s^\cdot} - Q_s\right] \delta q_s = 0 \tag{1.6}$$

$$T = T(\mathbf{q}, \mathbf{q}^\cdot, \pi, t), \quad \Pi = \Pi(\mathbf{q}, \pi, t), \quad R = R(\mathbf{q}, \mathbf{q}^\cdot, \pi, t)$$

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where T, Π and R respectively are the kinetic energy, the potential energy, and the Rayleigh scattering function.

Of the n virtual displacements δq_s, only $n-m$ are independent, because by (1.1) $\mathbf{a}^{\times}\delta\mathbf{q}=0$. Assume, to fix our ideas, that $\delta q_{m+1},\ldots,\delta q_n$ are the independent virtual displacements. Put $\mathbf{q}'=[q_{m+1}\ldots q_n]^T$ and $\mathbf{q}''=[q_1\ldots q_m]^T$. Then Eqs.(1.1) enable us to express $\mathbf{q}''^{\cdot}$ in terms of $\mathbf{q}'^{\cdot}$ and $\mathbf{q}$ in the form

$$\mathbf{q}''^{\cdot}=\mathbf{B}'\mathbf{q}'^{\cdot}+\mathbf{B}'' \tag{1.7}$$

$$\mathbf{B}'=\mathbf{B}'(\mathbf{q},\pi,t)\equiv\|B_{is}'\|,\quad \mathbf{B}''=\mathbf{B}''(\mathbf{q},\pi,t)\equiv\|B_i''\|;\quad i=1,\ldots,m;$$
$$s=m+1,\ldots,n$$

where $\mathbf{B}'$ is an $m\times(n-m)$ matrix and $\mathbf{B}''$ is an m-dimensional column matrix.

Substitute (1.7) into (1.6) and equate the coefficients of the independent variations δq_s to zero. Then, denoting by an asterisk the result obtained after eliminating the dependent generalized velocities $\mathbf{q}''^{\cdot}$ with the aid of (1.7), we obtain equations in Ito stochastic differentials

$$d\left(\frac{\partial T^*}{\partial q_s^{\cdot}}\right)+\left[-\frac{\partial(T^*-\Pi)}{\partial q_s}+\frac{\partial R^*}{\partial q_s^{\cdot}}-\sum_{i=1}^m B_{is}'\frac{\partial(T^*-\Pi)}{\partial q_i}-\right. \tag{1.8}$$
$$\left.\left(Q_s+\sum_{i=1}^m B_{is}'Q_i\right)\right]dt-\sum_{i=1}^m\left\{\left(\frac{\partial T}{\partial q_i^{\cdot}}\right)^*\left[dB_{is}'-\sum_{\beta=m+1}^n\left(\frac{\partial B_{i\beta}'}{\partial q_s}+\right.\right.\right.$$
$$\left.\left.\sum_{\varkappa=1}^m B_{\varkappa s}'\frac{\partial B_{i\beta}'}{\partial q_\varkappa}\right)dq_\beta-\left(\frac{\partial B_i''}{\partial q_s}+\sum_{\varkappa=1}^m B_{\varkappa s}'\frac{\partial B_i''}{\partial q_\varkappa}\right)dt\right]+\Delta\sigma_{is}^*=0$$
$$\Delta\sigma_{is}=d(U_1U_2)-U_1dU_2-U_2dU_1,\quad U_1\equiv dT/\partial q_i,\ U_2\equiv B_{is}'$$

In (1.8), the total differential of the compound function $u=u(\mathbf{z},\pi,t)$ is evaluated by the generalized Ito formula /1/

$$du=\frac{\partial u}{\partial t}dt+\frac{\partial^T u}{\partial \mathbf{z}}d\mathbf{z}+D_1[u]\,dt+\mathbf{D}_2[u]\,d\mathbf{W}_0+\int_{R^l}\mathbf{D}_3[u]\,d\mathbf{P}^\circ(t,d\mathbf{x}) \tag{1.9}$$
$$D_1[u]=\frac{\partial^T u}{\partial \pi}\varphi+\frac{1}{2}\operatorname{tr}(u_{\pi\pi}\Psi_0)+$$
$$\int_{R^l}\left[u(\mathbf{z},\pi+\psi\mathbf{c},t)-u(\mathbf{z},\pi,t)-\frac{\partial^T u}{\partial\pi}\psi\mathbf{c}\right]\nu_P(t,\mathbf{x})\,d\mathbf{x}$$
$$\mathbf{D}_2[u]=\frac{\partial^T u}{\partial\pi}\psi,\quad \mathbf{D}_3[u]=u(\mathbf{z},\pi+\psi\mathbf{c},t)-u(\mathbf{z},\pi,t)$$
$$\Psi_0=\psi\nu_0\psi^T,\quad \mathbf{c}=\mathbf{c}(\mathbf{x}),\quad u_{\pi\pi}=\frac{\partial}{\partial\pi}\frac{\partial^T}{\partial\pi}u$$

Non-holonomic systems described by the system of stochastic differential Eqs.(1.2), (1.7) and (1.8) will be called normal stochastic non-holonomic systems if $\mathbf{W}(t)$ is a Wiener process, Poisson stochastic non-holonomic systems if $\mathbf{W}(t)$ is a Poisson process, and general-form stochastic non-holonomic systems if $\mathbf{W}(t)$ is an arbitrary process with independent increments. Eqs.(1.8) can be transformed using the generalized Ito formula (1.9) to the form

$$\sum_{j=m+1}^n g_{sj}\,dq_j^{\cdot}+\sum_{j,\beta=m+1}^n\left(\sum_{r=1}^3\gamma_{sj\beta}^r\right)q_j^{\cdot}\,dq_\beta+\sum_{j=m+1}^n\left(\sum_{r=1}^4\alpha_{sj}^r\right)dq_j- \tag{1.10}$$
$$[(F_s^1+Q_s^{11})\,dt+(\mathbf{F}_s^2+\mathbf{Q}_s^{21})\,d\mathbf{W}_0]-\int_{R^l}\mathbf{F}_s^3\,d\mathbf{P}^\circ(t,d\mathbf{x})=0$$

We will use the following representations and notation:

$$T^*-\Pi={}^1/_2\mathbf{q}'^{\cdot T}\mathbf{G}\mathbf{q}'^{\cdot}+\mathbf{G}_1\mathbf{q}'^{\cdot}-\Pi_0,\quad \mathbf{G}\equiv\|g_{sj}\|,\quad \mathbf{G}_1\equiv\|g_s^1\| \tag{1.11}$$
$$R^*={}^1/_2\mathbf{q}'^{\cdot T}\mathbf{A}\mathbf{q}'^{\cdot}+\mathbf{A}_0\mathbf{q}'^{\cdot}+A_0,\quad \mathbf{A}\equiv\|\alpha_{sj}^2\|,\quad \mathbf{A}_1\equiv\|\alpha_s\|$$
$$\left(\frac{\partial T}{\partial q_i^{\cdot}}\right)^*\equiv\sum_{j=m+1}^n\theta_{ij}q_j^{\cdot}+\theta_i,\quad -\sum_{i=1}^m\Delta\sigma_{is}^*\equiv\sum_{j=m+1}^n\sigma_{js0}^1\,dq_j+\sigma_s^1\,dt+\boldsymbol{\sigma}_s^2\,d\mathbf{W}_0+$$
$$\int_{R^l}\boldsymbol{\sigma}_s^3\,d\mathbf{P}^\circ(t,d\mathbf{x})+\sum_{j=m+1}^n q_j^{\cdot}\left[\boldsymbol{\sigma}_{js0}^2\,d\mathbf{W}_0+\int_{R^l}\boldsymbol{\sigma}_{js0}^3\,d\mathbf{P}^\circ(t,d\mathbf{x})\right]$$
$$F_s^1=-\frac{\partial\Pi_0}{\partial q_s}-\sum_{i=1}^m B_{is}'\frac{\partial\Pi_0}{\partial q_i}-\alpha_s+\sum_{i=1}^m\theta_i\rho_{si}+f_s^1$$

$$\mathbf{F}_s^2 = \mathbf{f}_s^2 - \sum_{j=m+1}^{n} \boldsymbol{\alpha}_{sj}^5 q_j^{\cdot}, \quad \mathbf{F}_s^3 = \mathbf{c}(\mathbf{x})\,\mathbf{f}_s^2 - \sum_{i=1}^{m} \theta_i \mathbf{D}_3 [B_{is}'] - \sum_{j=m+1}^{n} \boldsymbol{\alpha}_{sj}^6 q_j^{\cdot} - \boldsymbol{\sigma}_s^3$$

$$\mathbf{f}_s^k = \mathbf{Q}_s^k + \sum_{i=1}^{m} B_{is}' \mathbf{Q}_i^k, \quad \mathbf{Q}_s^{k1} = \sum_{i=1}^{m} Q_i \mathbf{D}_k [B_{is}'] - \boldsymbol{\sigma}_s^k, \quad k = 1, 2$$

$$\gamma_{sj\beta}^1 = \frac{1}{2}\left(\frac{\partial g_{js}}{\partial q_\beta} + \frac{\partial g_{s\beta}}{\partial q_j} - \frac{\partial g_{j\beta}}{\partial q_s}\right), \quad \gamma_{sj\beta}^2 = -\sum_{i=1}^{m} \theta_{ij} \gamma_{is\beta}^1,$$

$$\gamma_{is\beta}^3 = -\frac{1}{2}\sum_{i=1}^{m} B_{is}' \frac{\partial g_{j\beta}}{\partial q_i}$$

$$\gamma_{is\beta}' = \frac{\partial B_{is}'}{\partial q_\beta} - \frac{\partial B_{i\beta}'}{\partial q_s} + \sum_{\varkappa=1}^{m}\left(B_{\varkappa\beta}' \frac{\partial B_{is}'}{\partial q_\varkappa} - B_{\varkappa s}' \frac{\partial B_{i\beta}'}{\partial q_\varkappa}\right) = -\gamma_{i\beta s}'$$

$$\alpha_{sj}^1 = \frac{\partial g_s^1}{\partial q_j} - \frac{\partial g_j^1}{\partial q_s} = -\alpha_{js}^1, \quad \alpha_{sj}^3 = -\sum_{i=1}^{m}\left(B_{is}' \frac{\partial g_s^1}{\partial q_i} + \theta_i \gamma_{isj}' + \theta_{ij}\rho_{sj}\right)$$

$$\rho_{sj} = \frac{\partial B_{is}'}{\partial t} - \frac{\partial B_i''}{\partial q_s} + \sum_{\varkappa=1}^{m}\left(B_\varkappa'' \frac{\partial B_{is}'}{\partial q_\varkappa} - B_{\varkappa s}' \frac{\partial B_i''}{\partial q_\varkappa}\right)$$

$$\alpha_{sj}^4 = \beta_{sj}^1 + \sigma_{js0}^1, \quad \alpha_{sj}^5 = \beta_{sj}^2 + \sigma_{js0}^2, \quad \alpha_{sj}^6 = \beta_{sj}^3 + \sigma_{js0}^1$$

$$\boldsymbol{\beta}_{sj}^k = -\sum_{i=1}^{m} \theta_{ij} \mathbf{D}_k [B_{is}'] + \mathbf{D}_k [g_{sj}], \quad k = 1, 2, 3$$

As the initial values for $t = t_0$ in (1.10) we take the random variables $q_i = q_{i0}$, $q_i^{\cdot} = q_{i0}^{\cdot}$. For a normal stochastic non-holonomic system, when $\mathbf{W} = \mathbf{W}_0$, we have

$$\mathbf{D}_3[u] = 0, \quad \mathbf{v}_P = 0, \quad \boldsymbol{\sigma}_{js0}^3 = \boldsymbol{\sigma}_s^3 = 0, \quad \mathbf{F}_s^3 = 0$$

and Eq.(1.10) may be represented in the form of Ito stochastic differential equations

$$\sum_{j=m+1}^{n} g_{sj} q_j^{\cdot\cdot} + \sum_{j,\beta=m+1}^{n} \left(\sum_{r=1}^{3} \gamma_{sj\beta}^r\right) q_j^{\cdot} q_\beta^{\cdot} + \sum_{j=m+1}^{n} \left(\sum_{r=1}^{4} \boldsymbol{\alpha}_{sj}^r + \boldsymbol{\alpha}_{sj}^5 \mathbf{V}_0\right) q_j^{\cdot} - \\ [F_s^1 + Q_s^{11} + (\mathbf{F}_s^2 + \mathbf{Q}_s^{21}) \mathbf{V}_0] = 0 \quad (\mathbf{V}_0 = d\mathbf{W}_0/dt) \tag{1.12}$$

Analysis of the structure of the equations of motion (1.12) leads to the following conclusions. The terms with second derivatives characterize inertial forces. The terms quadratic in velocity characterize gyroscopic forces for $r = 1$ and 2 and some generally dissipative forces for $r = 3$. The terms linear in velocity define gyroscopic forces for $r = 1$, dissipative forces for $r = 2$, forces caused by non-homogeneity and non-autonomy of the constraints for $r = 3$, perturbing forces that vanish in the absence of random perturbations for $r = 4$, and irregular perturbing forces for $r = 5$. The expression in square brackets characterizes a certain equivalent perturbing force, where $Q_s^{11} = 0$ and $\mathbf{Q}_s^{21} = 0$ when there are no random perturbations ($\pi = 0$).

Note that in the important practical case when the coefficients g_{sj}, g_s^1 in the expression for the kinetic energy and the coefficients B_{is}' of the non-holonomic constraints are linear functions of the random parameters $\boldsymbol{\pi}$, we have $\Delta\sigma_{is} \equiv 0$, and Eqs.(1.10) of the general-form stochastic non-holonomic system can also be reduced to the form (1.12). The equations of motion of a Poisson stochastic non-holonomic systems have a similar form.

The equations of a normal stochastic non-holonomic system can be rewritten in the Hamiltonian form:

$$\mathbf{q}'^{\cdot} = \partial H/\partial \mathbf{p}' \equiv \mathbf{Y}(\mathbf{q}, \mathbf{p}', \pi, t) \\ \mathbf{p}'^{\cdot} = \mathbf{U}(\mathbf{q}, \mathbf{p}', \pi, t) + \mathbf{U}''(\mathbf{q}, \mathbf{p}', \pi, t)\,\mathbf{V} \tag{1.13}$$

where $\mathbf{p}' = \partial T^*/\partial \mathbf{q}'^{\cdot}$, H s the Legendre transform of the function $L^* = T^* - \Pi^*$. and $dp_s' - U_s dt - \mathbf{U}_s'' d\mathbf{W}_0 = 0$ is the equivalent form of Eq.(1.8) with $\mathbf{q}'^{\cdot}$ replaced everywhere by $\mathbf{Y}$. These equations should be supplemented by the relationship

$$\mathbf{q}''^{\cdot} = \mathbf{B}'\mathbf{Y} + \mathbf{B}'' \equiv \boldsymbol{\Lambda}(\mathbf{q}, \mathbf{p}', \pi, t) \tag{1.14}$$

Combining Eqs.(1.2), (1.13) and (1.14) with the corresponding initial conditions, we represent them by the following equation for the augmented state vector $\mathbf{Z} = [\boldsymbol{\pi}^T \mathbf{q}^T \mathbf{p}'^T]^T$:

$$\mathbf{Z}^{\cdot} = \mathbf{a}(\mathbf{Z}, t) + \mathbf{b}(\mathbf{Z}, t)\,\mathbf{V}_0, \quad \mathbf{Z}(t_0) = \mathbf{Z}_0 \tag{1.15}$$

$$\mathbf{a}(\mathbf{Z},t)=\begin{Vmatrix}\varphi(\pi,t)\\ \mathbf{X}(\pi,\mathbf{q},\mathbf{p}',t)\\ \mathbf{U}(\pi,\mathbf{q},\mathbf{p}',t)\end{Vmatrix},\quad \mathbf{b}(\mathbf{Z},t)=\begin{Vmatrix}\psi(\pi,t)\\ 0\\ \mathbf{U}''(\pi,\mathbf{q},\mathbf{p}',t)\end{Vmatrix},\quad \mathbf{X}=\begin{Vmatrix}\Lambda(\pi,\mathbf{q},\mathbf{p}',t)\\ \Upsilon(\pi,\mathbf{q},\mathbf{p}',t)\end{Vmatrix}$$

2. Equations for the distributions of the state vector of a stochastic non-holonomic system described by Ito stochastic differential Eqs.(1.15) can be obtained by various exact and approximate methods of the theory of stochastic differential systems (see, e.g., /1/).

Let us write the equations for the characteristic functions and the densities of the state vector $\mathbf{Z}(t)$ assuming the one-dimensional distribution (and therefore all the finite-dimensional distributions) of the process with independent increments $\mathbf{W}(t)$ defined by (1.3) to be known. Denote by $h_1(\rho;t)$ and $g_1(\lambda;t)$ the one-dimensional characteristic functions of the processes $\mathbf{W}(t)$ and $\mathbf{Z}(t)$, respectively. Then the characteristic function $g_1(\lambda;t)$ satisfies the well-known Pugachev equation

$$\begin{gathered}\partial g_1(\lambda;t)/\partial t=M\{i(\lambda^\pi)^T\varphi(\pi,t)+i(\lambda^q)^T\mathbf{X}(\pi,\mathbf{q},\mathbf{p}',t)+i(\lambda^{p'})^T\mathbf{U}(\pi,\mathbf{q},\mathbf{p}',t)+\\ \chi[\psi(\pi,t)^T\lambda^\pi;t]+\chi[\mathbf{U}''(\pi,\mathbf{q},\mathbf{p}',t)^T\lambda^{p'};t]\}\exp(\iota\lambda^T\mathbf{Z})\\ \lambda=[(\lambda^\pi)^T(\lambda^q)^T(\lambda^{p'})^T]^T,\quad \chi(\rho;t)\equiv\partial\ln h_1(\rho;t)/\partial t,\quad \iota=\sqrt{-1}\end{gathered}\tag{2.1}$$

with the initial condition $g_1(\lambda;t_0)=g_0(\lambda)$, where $g_0(\lambda)$ is the characteristic function of $\mathbf{Z}(t_0)$. The equation and the initial condition (2.1) under certain conditions completely and uniquely define $g_1(\lambda;t)$ for $t\geqslant t_0$ /1/.

Followig /1/, we can write, as for (2.1), the equations for the n-dimensional $(n=2,3,\ldots)$ characteristic function of the process $\mathbf{Z}=\mathbf{Z}(t)$.

The specific form of the function $\chi(\rho;t)$ in Eq.(2.1) is determined by the nature of the process $\mathbf{W}(t)$. If $\mathbf{W}(t)$ is a Wiener process, then $\chi(\rho;t)=-1/2\rho^T\mathbf{v}(t)\rho$; if $\mathbf{W}(t)$ is a general Poisson process, then $\chi(\rho;t)=[g(\rho)-1]\mathbf{v}(t)$, where $g(\rho)$ is the characteristic function of the jumps. If the process $\mathbf{W}(t)$ consists of N independent blocks, $\mathbf{W}(t)=[\mathbf{W}_1(t)^T\ldots\mathbf{W}_N(t)^T]^T$, then partitioning ρ into corresponding blocks, we have

$$\chi(\mathbf{b}(\mathbf{z},t)^T\lambda;t)=\sum_{k=1}^N\chi_k(\mathbf{b}_k(\mathbf{z},t)^T\lambda;t),\ \mathbf{b}(\mathbf{z},t)=[\mathbf{b}_1(\mathbf{z},t)\ldots\mathbf{b}_N(\mathbf{z},t)]$$

In general, when the process $\mathbf{W}(t)$ is defined by (1.3), we obtain for the function $\chi(\rho;t)$ and for the mean number of c(x) jumps of the process $\mathbf{W}(t)$

$$\chi(\rho;t)=-\frac{1}{2}\rho^T\mathbf{v}_0(t)\rho+\int_{R^l}[\exp(i\rho^T\mathbf{c}(\mathbf{x}))-1-i\rho^T\mathbf{c}(\mathbf{x})]\mathbf{v}_P(t,\mathbf{x})\,d\mathbf{x}$$

$$\mu(t,\mathbf{x})\,d\mathbf{x}=\int_0^t\mathbf{v}_P(\tau,\mathbf{x})\,d\tau\,d\mathbf{x}$$

For normal white noise $\mathbf{V}$, Eq.(2.1) leads to the Fokker-Planck-Kolmogorov equation for the one-dimensional density $f_1=f_1(\pi,\mathbf{q},\mathbf{p}',t)$

$$\begin{gathered}\frac{\partial f_1}{\partial t}=-\frac{\partial^T}{\partial\pi}(\varphi f_1)-\frac{\partial^T}{\partial\mathbf{q}'}(\mathbf{X}f_1)-\frac{\partial^T}{\partial\mathbf{p}'}(\mathbf{U}f_1)+\\ \frac{1}{2}\operatorname{tr}\left[\frac{\partial}{\partial\pi}\frac{\partial^T}{\partial\pi}(\psi\mathbf{v}\psi^Tf_1)\right]+\frac{1}{2}\operatorname{tr}\left[\frac{\partial}{\partial\mathbf{p}'}\frac{\partial^T}{\partial\mathbf{p}'}(\mathbf{U}''\mathbf{v}\mathbf{U}''^Tf_1)\right]\end{gathered}\tag{2.2}$$

For a stationary normal stochastic non-holonomic system, the one-dimensional stationary distribution is determined from (2.1) for $\partial g_1/\partial t=0$ or from (2.2) for $\partial f_1/\partial t=0$.

In the general case, when the process $\mathbf{W}=\mathbf{W}(t)$ is defined by formula (1.3), the corresponding equation for the one-dimensional density is given in /1, p.314/.

Remark. The equations for the distributions of the state vector of a normal stochastic non-holonomic system (1.15) can be obtained by assuming stochastic processes in Stratonovich's sense in the initial equations of the problem and at the last stage applying Ito's formula (1.9) for the Wiener process $\mathbf{W}=\mathbf{W}_0$. For Poisson systems and general-form systems, this technique does not work, because the corresponding stochastic process is not Markov.

3. The equations of motion of the stochastic non-holonomic system (1.1), (1.6) can be obtained using the general theorems of dynamics or, for instance, the Appel equation. If the

resulting system of stochastic equations of motion is linear, then we can directly write explicit formulas for finite-dimensional distributions.

Example. Consider the rolling of a homogeneous ball on a translationally vibrating plane with rolling resistance forces. The vibrational acceleration is an arbitrary vector white noise $V = [V_x V_y]^T$ with constant intensities ν_x, ν_y. In what follows, x and y are the coordinates of the centre of the ball relative to the vibrating plane. Then, for $u_x = x^\cdot$ and x we obtain the equations

$$u_x^\cdot + 2\varepsilon u_x = \delta V_x, \quad x^\cdot = u_x, \quad \delta = \text{const} \tag{3.1}$$

($\varepsilon > 0$ is the coefficient of friction). The equations for u_y and y are obtained from (3.1) by replacing x with y. A kinematic constraint links the projections of the angular velocity ω on the axes x and y and the components u_x, u_y: $u_x = r\omega_y$, $u_y = -r\omega_x$ (r is the radius of the ball). The stochastic Eqs.(3.1) are linear Langevin equations. With zero initial conditions, the means, variances, and covariances of the processes u_x, x are determined by the well-known formulas /1, p.307/, which give

$$k_{xx} \approx 0, \quad k_{xu_x} \approx 0, \quad k_{u_x u_x} \approx \nu_x \delta^2 t \quad (t \sim 0)$$
$$k_{xx} \approx \nu_x \delta^2 t/(4\varepsilon^2), \quad k_{xu_x} \approx \nu_x \delta^2/(8\varepsilon^2), \quad k_{u_x u_x} \approx \nu_x \delta^2/(4\varepsilon) \quad (t \to \infty)$$

Explicit formulas for the finite-dimensional characteristic functions in this case are given in /1/.

4. We know that in non-linear mechanical systems under the action of perturbing forces represented by normally distributed white noise there exists a narrow-sense stationary process, whose one-dimensional density is given by the Gibbs formula /3/. A generalization of the Gibbs distribution has been obtained for some types of holonomic systems /1, 4, 5/. The structure of the equations of the non-holonomic systems (1.13), (1.14) suggests the following generalization.

Proposition 1. Assume that the stochastic equations of motion of some mechanical system in canonical variables have the form

$$\mathbf{q}'^\cdot = \frac{\partial H}{\partial \mathbf{p}'}, \quad \mathbf{p}'^\cdot = -\frac{\partial H}{\partial \mathbf{q}'} - 2\varepsilon \mathbf{a}(\mathbf{q}')\frac{\partial H}{\partial \mathbf{p}'} + \mathbf{b}(\mathbf{q}')\mathbf{V}, \quad \mathbf{q}''^\cdot = \mathbf{\Lambda}(\mathbf{q}', \mathbf{p}')$$

Here 2ε is the specific coefficient of viscous friction, $\mathbf{V}$ is the vector of independent normally distributed white noises of the same constant intensity ν; $\mathbf{a} = \mathbf{a}(\mathbf{q}')$ and $\mathbf{b} = \mathbf{b}(\mathbf{q}')$ are matrix functions. Let $\mathbf{x} = [\mathbf{q}'^T \mathbf{p}'^T]^T = [\mathbf{x}'^T \mathbf{x}''^T]^T$ $\dim \mathbf{x}' = \dim \Lambda \leqslant \dim \mathbf{x}$. If

$$1)\ \mathbf{a} + \mathbf{a}^T = 2\mathbf{b}\mathbf{b}^T, \quad 2)\ |\partial \mathbf{\Lambda}/\partial \mathbf{x}'| \neq 0$$
$$3)\ \iint_{-\infty}^{\infty} \exp[-\alpha H(\mathbf{q}', \mathbf{p}')]\, d\mathbf{q}'\, d\mathbf{p}' < \infty \quad (\alpha = 4\varepsilon/\nu)$$

then there exists a narrow-sense stationary solution, and the one-dimensional densities $\mathbf{q}'$, $\mathbf{p}'$ and $\mathbf{p}'' = \mathbf{q}''^\cdot$ are given by the formulas

$$f_1(\mathbf{q}', \mathbf{p}') = c \exp[-\alpha H(\mathbf{q}', \mathbf{p}')] \tag{4.1}$$
$$f_1(\mathbf{p}'') = \int_{-\infty}^{\infty} f_1[\mathbf{\Lambda}^{-1}(\mathbf{p}'', \mathbf{x}''), \mathbf{x}'']\, J(\mathbf{p}'', \mathbf{x}'')\, d\mathbf{x}''$$
$$J(\mathbf{p}'', \mathbf{x}'') = |\partial \mathbf{\Lambda}^{-1}(\mathbf{p}'', \mathbf{x}'')/\partial \mathbf{p}''| \neq 0$$

where c is a constant determined from the normalization conditions.

The proof of Proposition 1 is similar to /1, Example 5.16/, using the formula for the distribution density of a function of a random argument /6/.

Proposition 1 can be generalized to non-holonomic systems of the form

$$\mathbf{q}'^\cdot = \mathbf{Y}(\mathbf{q}', \mathbf{p}'), \mathbf{p}'^\cdot = \mathbf{U}_1(\mathbf{q}', \mathbf{p}') - 2\varepsilon \mathbf{U}_2(\mathbf{q}', \mathbf{p}') + \mathbf{b}(\mathbf{q}')\mathbf{V}, \quad \mathbf{q}''^\cdot = \mathbf{\Lambda}(\mathbf{q}', \mathbf{p}') \tag{4.2}$$

which without friction ($\varepsilon = 0$) and without random perturbations ($\mathbf{b} = 0$) have an invariant measure with some first integral $H = H(\mathbf{q}', \mathbf{p}')$.

Proposition 2. Assume that for $\varepsilon = 0$, $\mathbf{b} = 0$, system (4.2) has an invariant measure with density $N(\mathbf{q}')$, i.e., $\partial^T(N\mathbf{Y})/\partial \mathbf{q}' + \partial^T(N\mathbf{U}_1)/\partial \mathbf{p}' = 0$, and the first integral $H = H(\mathbf{q}', \mathbf{p}')$. If $\mathbf{U}_2 = \mathbf{a}(\mathbf{q}')\partial H/\partial \mathbf{p}'$ and conditions 1-3 of Proposition 1 are satisfied, then there exists a narrow-sense stationary solution and the one-dimensional densities $\mathbf{q}'$, $\mathbf{p}'$, and $\mathbf{p}'' = \mathbf{q}''^\cdot$ are given by the formulas

$$f_1(\mathbf{q}', \mathbf{p}') = cN(\mathbf{q}') \exp[-\alpha H(\mathbf{q}', \mathbf{p}')], \quad \alpha = 4\varepsilon/\nu \tag{4.3}$$

and (4.1), respectively.

Propositions 1 and 2 provide exact stationary solutions only for normal non-holonomic systems that satisfy the required conditions.

For a number of general-form systems encountered in statistical non-holonomic non-linear mechanics, the finite-dimensional distributions can be approximately obtained by the normal approximation method /1/.

5. In what follows, some of the notation from Sects.1-4 will be used in a different sense, as conventionally accepted in the dynamics of rigid bodies. Consider a heavy rigid body enclosed by a strictly convex surface and resting on an absolutely rough horizontal surface. We assume that the support surface performs translational vibration. The position of the body is defined by the coordinates $r = [xyz]^T$ of its centre of mass G in the coordinates system $Oxyz$ (the plane Oxy coincides with the vibrating support plane, and the Oz axis points vertically up), and the Euler angles $q = [\psi\varphi\theta]^T$ that define the orientation of the principal central axes of inertia $G\xi\eta\zeta$ of the body relative to the coordinates system $Oxyz$. We introduce a fixed system of coordinate $O_1x_1y_1z_1$ with axes parallel to the corresponding axes of the moving coordinate systems $Oxyz$. The position of the vibrating plane is given if we know the time variation of the vector $\mathbf{O_1O}(t)$. The components of the vector $\mathbf{O_1O} = [x_0(t) y_0(t) z_0(t)]^T$ are treated as independent stochastic processes. We assume that the body moves without separating from the support surface, so that $z = z(\varphi, \theta)$.

The constraint equations of the system (expressing the absence of sliding at the point of contact of the body with the plane) are deterministic and have the form /7/

$$\mathbf{r}^{\cdot} = \mathbf{b}^T(\mathbf{q})\mathbf{q}^{\cdot}, \quad \mathbf{b} = \| b_{ij} \| \quad (i, j = 1, 2, 3) \tag{5.1}$$
$$b_{11} = -z_\varphi \sin\psi/\sin\theta - z_\theta\cos\psi, \quad b_{12} = -\partial b_{11}/\partial\psi, \quad b_{13} = 0$$
$$b_{21} = -z_\varphi \sin\psi \operatorname{ctg}\theta - (z_\theta\cos\theta + z\sin\theta)\cos\psi,$$
$$b_{22} = -\partial b_{21}/\partial\psi, \quad b_{23} = z_\varphi$$
$$b_{31} = z\sin\psi, \quad b_{32} = -z\cos\psi, \quad b_{33} = z_\theta \quad (z_\theta = \partial z/\partial\theta, \; z_\varphi = \partial z(\partial\varphi))$$

Note that $\mathbf{b}_1 = \mathbf{b}\mathbf{b}^T$ is a matrix function of the angles φ and θ only.

The Lagrangian function of the system has the form

$$L = L_1 + m(x_0^{\cdot 2} + y_0^{\cdot 2} + z_0^{\cdot 2} + 2x^{\cdot}x_0^{\cdot} + 2y^{\cdot}y_0^{\cdot} + 2z^{\cdot}z_0^{\cdot})/2 \tag{5.2}$$
$$L_1 = m(x^{\cdot 2} + y^{\cdot 2})/2 + (A\cos^2\varphi + B\sin^2\varphi + mz_\theta^2)/\theta^{\cdot 2}/2 +$$
$$(C + mz_\varphi^2)\varphi^{\cdot 2}/2 + [(A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + C\cos^2\theta]\psi^{\cdot 2}/2 +$$
$$mz_\theta z_\varphi\theta^{\cdot}\varphi^{\cdot} + C\cos\theta\psi^{\cdot}\varphi^{\cdot} + (A - B)\sin\theta\sin\varphi\cos\varphi\theta^{\cdot}\psi^{\cdot} - mgz$$

where m is the mass of the body, A, B, C are its principal central moments of inertia, and L_1 is the Lagrangian function corresponding to the motion of the body on a fixed absolutely rough surface /7/.

Assume that dissipative forces with the Rayleigh function $\Phi = \Phi(\mathbf{r}^{\cdot}, \mathbf{q}, \mathbf{q}^{\cdot}, \mathbf{O_1O}^{\cdot})$ act on the body. The motion of a rigid body on a horizontal, absolutely rough, translationally vibrating plane can be reduced to the motion of a rigid body on a fixed horizontal absolutely rough plane in the presence of perturbing forces of the form $\mathbf{Q} = -m\mathbf{b}\mathbf{O_1O}^{\cdot\cdot}$.

Indeed, for the elementary work $\delta A'$ of the inertial forces in the moving coordinate system $Oxyz$ we have the expression

$$\delta A' = -m\mathbf{O_1O}^{\cdot\cdot T}\delta\mathbf{r} = -m\mathbf{O_1O}^{\cdot\cdot T}\mathbf{b}^T\delta\mathbf{q}$$

Hence $\mathbf{Q} = -m\mathbf{b}\mathbf{O_1O}^{\cdot\cdot}$.

Therefore, the equations of motion of the body may be written in the form

$$\frac{d}{dt}\left(\frac{\partial L_1^*}{\partial \mathbf{q}^{\cdot}}\right) - \frac{\partial L_1^*}{\partial \mathbf{q}} = \boldsymbol{\Gamma} - \frac{\partial\Phi}{\partial\mathbf{q}^{\cdot}} + \mathbf{Q} \tag{5.3}$$

where $\boldsymbol{\Gamma}$ is a column matrix of the non-holonomic terms, and L_1^*, Φ^* are obtained from L_1 and Φ by replacing $x^{\cdot}, y^{\cdot}, z^{\cdot}$ by the corresponding expressions (5.1) in terms of $\mathbf{q}$ and $\mathbf{q}^{\cdot}$. The explicit form of L_1^* and $\boldsymbol{\Gamma}$ is given in /7/.

Eqs.(5.3) can be written in Hamiltonian form as

$$\mathbf{q}^{\cdot} = \frac{\partial H}{\partial\mathbf{p}}, \quad \mathbf{p}^{\cdot} = -\frac{\partial H}{\partial\mathbf{q}} - \frac{\partial\Phi^*}{\partial\mathbf{q}^{\cdot}} + \boldsymbol{\Gamma} + \mathbf{Q}, \quad \mathbf{p} = [p_\psi p_\varphi p_\theta]^T = \frac{\partial L_1}{\partial\mathbf{q}^{\cdot}} \tag{5.4}$$

Here H is the Legendre transform of the function L_1^*;on the right-hand side of the equations for $\mathbf{p}^{\cdot}$, all $\mathbf{q}^{\cdot}$ are replaced by $\partial H/\partial\mathbf{p}$.

Below we will consider two models of viscous friction. The first corresponds to the Rayleigh function $\Phi_1 = m\varepsilon \mathbf{U}_e^2$ and the second to $\Phi_2 = m\varepsilon \mathbf{U}^2$, where U and U_e are the absolute and the relative velocities (in relation to the moving coordinates system $Oxyz$) of the centre of mass G, and $\varepsilon > 0$ is the specific coefficient of friction.

$$\Phi_1^* = m\varepsilon \mathbf{q}^{\cdot T}\mathbf{b}_1\mathbf{q}^{\cdot}, \quad \Phi_2^* = \Phi_1^* + 2m\varepsilon \mathbf{O}_1\mathbf{O}^{\cdot T}\mathbf{b}^T\mathbf{q}^{\cdot} + m\varepsilon \mathbf{O}_1\mathbf{O}^{\cdot 2}$$

In the first case, we assume that the vibrational accelerations $\mathbf{O}_1\mathbf{O}^{\cdot\cdot}$ constitute a vector $\mathbf{V}$ of independent normally distributed white noise of constant intensity ν. In the second case, we treat $\pi = \mathbf{O}_1\mathbf{O}^{\cdot}$ as the vector of stationary random functions satisfying the shaping filter differential equation $\pi^{\cdot} + 2\varepsilon\pi = \mathbf{V}$ /1, p.260/. We can thus write the equations of motion (5.4) in both cases in the form of a system of stochastic non-linear differential equations

$$\mathbf{q}^{\cdot} = \frac{\partial H}{\partial \mathbf{p}}, \quad \mathbf{p}^{\cdot} = -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{\Gamma} - 2\varepsilon m\mathbf{b}_1\frac{\partial H}{\partial \mathbf{p}} - m\mathbf{b}\mathbf{V} \tag{5.5}$$

Eqs.(5.5) for $\varepsilon = 0$, $\mathbf{b} = 0$ admit of the energy integral $H = \text{const}$. Moreover, conditions 1-3 of Proposition 1 are satisfied. However, in general, these equations do not have an invariant measure /8/.

Let us consider one of the cases when an invariant measure nevertheless exists. Assume that the body is enclosed by a surface of revolution with the axis ζ and it has dynamic symmetry, i.e., $A = B$, $z = z(\theta)$. Then $H, \Gamma, \mathbf{b}_1$ are independent of the angles ψ and φ Eqs. (5.5) for $\varepsilon = 0$, $\mathbf{b} = 0$ admit of three first integrals /9/ and have an invariant measure with density $N(\theta)$ /10/:

$$N(\theta) = \delta^{-1}(\theta), \quad \delta(\theta) = (1 + mC^{-1}\mu^2 + mA^{-1}\varkappa^2)^{1/2}$$
$$\mu = z\sin\theta + z_\theta\cos\theta, \quad \varkappa = -z\cos\theta + z_\theta\sin\theta$$

Only the energy integral $H = H(\mathbf{p}, \theta)$ satisfies condition 3 of Proposition 1. Therefore, by Proposition 2, a stationary solution exists and the one-dimensional density for the variables θ, p_θ, p_ψ and p_φ is given by the formula

$$f_1(\theta, p_\theta, p_\psi, p_\varphi) = cN(\theta)\exp(-2\varepsilon\nu^{-1}H) =$$
$$cN(\theta)\exp\{-2\varepsilon\nu^{-1}[(A + m(z^2 + z_\theta^2))^{-1}p_\theta^2 + (C + m\mu^2)\Delta^{-1}p_\psi^2 -$$
$$(C\cos\theta + mz_\theta\mu)\Delta^{-1}p_\psi p_\varphi + (A\sin^2\theta + C\cos^2\theta + mz_\theta^2)\Delta^{-1}p_\varphi^2]\},$$
$$\Delta = AC\delta^2\sin^2\theta$$

6. An important special case of normal stochastic non-holonomic systems is Chaplygin normal stochastic non-holonomic systems which satisfy the following conditions:

$$\mathbf{a}^{\varkappa}(\mathbf{q}, \pi, t) = \mathbf{a}^{\varkappa}(\mathbf{q}', \pi, t), \quad a_0^{\varkappa}(\mathbf{q}, \pi, t) = a_0^{\varkappa}(\mathbf{q}', \pi, t) \tag{6.1}$$

$$Q_s^1(\mathbf{q}, \mathbf{q}^{\cdot}, \pi, t) = Q_s^1(\mathbf{q}', \mathbf{q}^{\cdot}, \pi, t), \; Q_s^2(\mathbf{q}, \mathbf{q}^{\cdot}, \pi, t) = Q_s^2(\mathbf{q}', \mathbf{q}^{\cdot}, \pi, t) \tag{6.2}$$
$$(s = m+1, \ldots, n), \quad Q_1 = \ldots = Q_m = 0$$

$$T(\mathbf{q}, \mathbf{q}^{\cdot}, \pi, t) = T(\mathbf{q}', \mathbf{q}^{\cdot}, \pi, t), \quad \Pi(\mathbf{q}, \pi, t) = \Pi(\mathbf{q}', \pi, t) \tag{6.3}$$
$$R(\mathbf{q}, \mathbf{q}^{\cdot}, \pi, t) = R(\mathbf{q}', \mathbf{q}^{\cdot}, \pi, t), \quad \mathbf{q}' = [q_{m+1} \cdots q_n]^T$$

The equations of motion of a Chaplygin non-holonomic stochastic system in Lagrange variables have the form (1.12) with

$$\gamma_{sj\beta}^3 = 0, \quad \gamma_{is\beta}' = \frac{\partial B_{is}'}{\partial q_\beta} - \frac{\partial B_{i\beta}'}{\partial q_s} = -\gamma_{i\beta s}', \quad \alpha_{sj}^3 = -\sum_{i=1}^{m}(\theta_i\gamma_{isj}' + \theta_{ij}\rho_{sj}) \tag{6.4}$$
$$F_s' = -\frac{\partial \Pi_0}{\partial q_s} - \alpha_s + \sum_{i=1}^{m}\theta_i\rho_{si} + f_s^1, \quad D_1[u] = \frac{\partial^T u}{\partial \boldsymbol{\pi}}\boldsymbol{\varphi} + {}^1/_2\,\mathrm{tr}(u_{\pi\pi}\boldsymbol{\Psi}_0)$$

(the expressions for the other quantities are given in (1.9) and (1.11)).

The stochastic equations of motion of a Chaplygin non-holonomic stochastic system in canonical Hamiltonian variables reduce to the form (1.3), (1.14) for

$$H = \frac{1}{2}\mathbf{p}'^T\mathbf{G}^{-1}\mathbf{p}' - \mathbf{G}_1\mathbf{G}^{-1}\mathbf{p}' + \frac{1}{2}\mathbf{G}_1\mathbf{G}^{-1}\mathbf{G}_1^T + \Pi_0 \tag{6.5}$$
$$\mathbf{Y} = \mathbf{G}^{-1}\mathbf{p}' - \mathbf{G}^{-1}\mathbf{G}_1^T, \quad \mathbf{\Lambda} = \mathbf{B}'\mathbf{Y} + \mathbf{B}''$$

Since the stochastic equations are decomposed into two blocks for the Lagrange variables $q', q^{\cdot}$ (or q', p') and for the variables q'' the solution sequence is obvious /11/. We should first find the distributions of the variables $q', q'^{\cdot}$ (or q', p'), and then apply the formulas for the distribution of a function of a random argument to find the distributions of $q''^{\cdot}$.

In the general case, because of non-linearity of the problem, it is impossible to obtain an exact solution of the equations for finite-dimensional distributions of $Z_1 = \pi, Z_2 = q', Z_3 = p', Z_4 = q''$ for normally distributed white noise V_0. The simplest approximate method of finding the finite-dimensional distributions of the state vector Z is the normal approximation method (NAM) /1/. The closer the system to a linear system, the higher the accuracy of the NAM results. Experience with the NAM in problems of applied mechanics shows, however, that it also produces good results for essentially non-linear systems.

We see from the structure of the functions (1.15) that there is no "restoring" force for the variables, and random fluctuations in the variables $q', q'^{\cdot}$ (or q', p') may produce drift. This is a qualitatively new effect, typical of stochastic non-holonomic systems.

Let us find the mean and the covariance function of the variable q'' assuming that we know the one-dimensional distribution $f_1(x; t)$ and the two-dimensional distribution $f_2(x', x''; t_1, t_2)$ for the vector $X = [Z_1^T Z_2^T Z_3^T]^T$.

From the third equation in (1.15) we obtain by integration

$$q'' = \int_{t_0}^{t} \Lambda(x, \tau)\, d\tau \tag{6.6}$$

Hence for the mean and the covariance function we obtain

$$m_{q''}(t) = \int_{t_0}^{t} m_\Lambda(\tau)\, d\tau, \quad m_\Lambda(t) = \int_{-\infty}^{\infty} \Lambda(x, t) f_1(x; t)\, dx \tag{6.7}$$

$$K_{q''}(t_1, t_2) = \int_{t_0}^{t_1}\int_{t_0}^{t_2} K_\Lambda(\tau_1, \tau_2)\, d\tau_1\, d\tau_2, \quad K_\Lambda(\tau_1, \tau_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Lambda^\circ(x', \tau_1)\Lambda^\circ(x'', \tau_2)^T \times$$
$$f_2(x', x''; \tau_1, \tau_2)\, dx'\, dx'', \quad \Lambda^\circ = \Lambda - m_\Lambda$$

The first two formulas in (6.7) define systematic drift, and the other formulas define fluctuational drift /12/.

Direct calculation of the statistical drift characteristics is quite difficult because of the non-linearity of non-holonomic problems. The computations are substantially simplified if, without sacrificing the non-linearity of the problem, we perform statistical linearization of the non-linear function Λ for the normal distribution

$$\Lambda(x, t) \approx \Lambda_0(m_x, K_x, t) + (\partial\Lambda_0(m_x, K_x, t)/\partial m_x)\, x^\circ$$
$$\Lambda_0 = M_N\Lambda, \quad x^\circ = x - m_x$$

As a result, formulas (6.7) take the form

$$m_{q''}(t) \approx \int_{t_0}^{t} \Lambda_0\, d\tau, \quad K_{q''}(t_1, t_2) \approx \int_{t_0}^{t_1}\int_{t_0}^{t_2} \frac{\partial\Lambda_0}{\partial m_x} K_x(\tau_1, \tau_2)\left(\frac{\partial\Lambda_0}{\partial m_x}\right)^T d\tau_1\, d\tau_2$$

For stationary fluctuations in x, setting $t_1 = t_2 = t,\ t_0 = 0,\ K_x(t_1, t_2) = k_x(t_1 - t_2)$, we obtain

$$m_{q''}(t) \approx \Lambda_0 t$$
$$K_{q''}(t) \approx \frac{\partial\Lambda_0}{\partial m_x}\left(\int_0^t \Lambda_1(\tau)\, d\tau\right)\left(\frac{\partial\Lambda_0}{\partial m_x}\right)^T, \quad \Lambda_1(\tau) = \int_0^\tau k_x(\xi)\, d\xi$$

The admissibility of the NAM is decided in the following way: if the spectral densities $s_x(\omega)$ and $s_{q''}(\omega)$ are close to one another at the frequency $\omega \sim 0$, the NAM can also be used to determine $K_{q''}$. If $s_x(\omega) \sim 0$, $\omega \sim 0$, then $s_{q''}(0) \neq 0$ and in this case fluctuational drift may not be ruled out. For ergodic broadband stochastic processes, fluctuational drift is always present, because the integral $\Lambda_1(\infty)$ exists and is non-zero. Conversely, for a narrow-bank process, $\Lambda_1(\infty)$ does not exist, and therefore there is no fluctuational drift /12/.

Remark. The equations for the mean of the state vector m, the covariance matrix K, and the covariance function $K(t_1, t_2)$ of the non-linear system (1.15) can be obtained by treating the stochastic processes in the original equations in Stratonovich's sense and applying Ito's formula only at the last stage to construct the equations for **m**, **K**, and $K(t_1, t_2)$ /1/.

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REFERENCES

1. PUGACHEV V.S. and SINITSYN I.N., Stochastic Differential Systems, Nauka, Moscow, 1985.
2. VORONETS P.V., On the equations of motion of non-holonomic systems, Mat. Sb., 22, 4, 1901.
3. GIBBS J., Thermodynamics. Statistical Mechanics, Nauka, Moscow, 1982.
4. FULLER A.T., Analysis of non-linear stochastic systems by means of the Fokker-Planck equation, Int. J. Control, 9, 6, 1969.
5. DIMENTBERG M.F., Stastical Dynamics of Non-Linear and Time-Varying Systems, Wiley, New York, 1988.
6. PUGACHEV V.S., Probability Theory and Mathematical Statistics, Nauka, Moscow, 1979.
7. KARAPETYAN A.V., On the permanent rotations of a heavy rigid body on an absolutely rough horizontal plane, PMM, 45, 5, 1981.
8. KOZLOV V.V., On the theory of integration of the equations of non-holonomic mechanics, Uspekhi Mekhaniki, 8, 3, 1985.
9. CHAPLYGIN S.A., Studies in the Dynamics of Non-holonomic Systems, Gostekhizdat, Moscow-Leningrad, 1949.
10. MOSHCHUK N.K., Qualitative analysis of the motion of a rigid body of revolution on an absolutely rough surface, PMM, 52, 2, 1988.
11. SINITSYN I.N. and SHIN V.I., Group-theoretical methods of decomposition in control problems for stochastic mechanical systems, Proceedings of the 5th All-Union Conf. on Control in Mechanical Systems, Izd. Kazan. Aviats. Inst., Kazan, 1985.
12. SINITSYN I.N., On the fluctuations of a gyroscope in a Cardan suspension, Izv. Akad Nauk SSSR, MTT, 3, 1976.

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